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THE PROBLEM OF A CIRCULAR PATCH†

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The problem of reinforcing an infinite thin elastic plate with a circular cut-out by means of a concentric circular patch attached to the plate so that the circumferences of the cut-out and the patch overlap is solved. The stress state in the plate and in the patch, generated by stresses at infinity in the plane of the plate, is investigated. Examples are presented.

1. FORMULATION OF THE PROBLEM

Consider a thin elastic plate with a circular cut-out, occupying in the complex plane $z = x + iy$ a domain $|z| \geq R_1$; and elastic circular patch $|z| \leq R_2$ $(R_1 < R_2)$ is placed on the plate and is attached to it with an overlap, without stretching or interlayers, along the circles L_1: $|z| = R_1$ and L_2: $|z| = R_2$ bordering the plate and the patch. The plate and the patch are divided by the curves L_1 and L_2, along which they are attached, into domains S_1: $R_1 < |z| < R_2$, S_2: $|z| > R_2$ and S_3: $|z| < R_1$, S_4: $R_1 < |z| < R_2$ respectively. The plate and the patch are uniform, isotropic and have thickness, shear modulus and Poisson's ratio h, μ, ν and h_0, μ_0, ν_0, respectively. Specified normal stresses σ_x^∞, σ_y^∞ and a shearing stress τ_{xy}^∞ are applied at infinity in the plane of the plate. The rotation at infinity is ω^∞. It will be assumed that the plate and patch surfaces touch one another without friction, that the space effects of stress concentration at the curves of attachment are negligibly small, and that the displacements of the points of these curves are equal and that the following equilibrium conditions are satisfied.

$$\begin{aligned}
&(u+iv)_1(t) = (u+iv)_3(t) = (u+iv)_4(t) \\
&h(X_n+iY_n)_1(t) + h_0(X_n+iY_n)_4(t) = h_0(X_n+iY_n)_3(t), \quad t \in L_1 \\
&(u+iv)_1(t) = (u+iv)_2(t) = (u+iv)_4(t) \\
&h(X_n+iY_n)_1(t) + h_0(X_n+iY_n)_4(t) = h(X_n+iY_n)_2(t), \quad t \in L_2
\end{aligned} \tag{1.1}$$

where $(u + iv)_k(t)$ is the displacement vector of a point t as viewed from the domain S_k and $(X_n + iY_n)_k(t)$ is the vector of stresses acting as viewed from S_k on an area element normal to the attachment curve at t, per unit thickness of the plate or the patch.

It is required to find the stress state of the above structure.

2. SOLUTION OF THE PROBLEM

The stresses and displacements in each of the domains S_k $(k = 1, 2, 3\ 4)$ are found using the well-known Kolosov–Muskhelishvili formulae [3] in terms of two functions $\varphi_k(z)$ and $\psi_k(z)$ analytic in S_k, which satisfy the following conditions on the circles L_1 and L_2 by virtue of (1.1)

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$$\mu_*(\kappa\varphi_1(t) + t\overline{\varphi_1'(t)} - \overline{\psi_1(t)}) = \kappa_0\varphi_k(t) - t\overline{\varphi_k'(t)} - \overline{\psi_k(t)}, \quad k = 3, 4$$

$$\varphi_1(t) + t\overline{\varphi_1'(t)} + \overline{\psi_1(t)} + h_* \sum_{k=3}^{4} (-1)^k(\varphi_k(t) + t\overline{\varphi_k'(t)} + \overline{\psi_k(t)}) = 0, \quad t = R_1 e^{i\theta}$$

$$\mu_*(\kappa\varphi_k(t) - t\overline{\varphi_k'(t)} - \overline{\psi_k(t)}) = \kappa_0\varphi_4(t) - t\overline{\varphi_4'(t)} - \overline{\psi_4(t)}, \quad k = 1, 2 \tag{2.1}$$

$$\sum_{k=1}^{2} (-1)^k(\varphi_k(t) + t\overline{\varphi_k'(t)} + \overline{\psi_k(t)}) - h_*(\varphi_4(t) + t\overline{\varphi_4'(t)} + \overline{\psi_4(t)}) = 0, \quad t = R_2 e^{i\theta}$$

$$0 \le \theta \le 2\pi, \quad \mu_* = \frac{\mu_0}{\mu}, \quad h_* = \frac{h_0}{h}, \quad \kappa = \frac{3-\nu}{1+\nu}, \quad \kappa_0 = \frac{3-\nu_0}{1+\nu_0}$$

For $k = 2, 3$ the functions $\varphi_k(z)$ and $\psi_k(z)$ are single-valued in the domain S_k and in the neighbourhood of infinity ,

$$\varphi_2(z) = \Gamma z + O(1), \quad \psi_2(z) = \Gamma' z + O(1)$$

$$\Gamma = \frac{1}{4}(\sigma_x^\infty + \sigma_y^\infty) + \frac{2i\mu}{1+\kappa}\omega^\infty, \quad \Gamma' = \frac{1}{2}(\sigma_y^\infty - \sigma_x^\infty) + i\tau_{xy}^\infty$$

but for $k = 1$ or $k = 4$ they have the following form in S_k

$$\varphi_k(z) = a_k \ln z + \varphi_k^*(z), \quad \psi_k(z) = -\kappa_k a_k \ln z + \psi_k^*(z), \quad k = 1, 4$$

where $\kappa_1 = \kappa$, $\kappa_4 = \kappa_0$ and $\varphi_k^*(z)$ and $\psi_k^*(z)$ are single-valued functions in S_k. Since the principal vectors of the forces applied at the attachment curve L_1 (or L_2) from the left and right, respectively, are equal, it follows that

$$h(1+\kappa)a_1 + h_0(1+\kappa_0)a_4 = 0 \tag{2.2}$$

Bearing the above properties in mind, we will seek the functions $\varphi_k(z)$ and $\psi_k(z)$ as series

$$\varphi_k(z) = a_k \ln z + \sum_{n=-\infty}^{\infty} c_{nk} z^n, \quad \psi_k(z) = -\kappa_k \bar{a}_k \ln z + \sum_{n=-\infty}^{\infty} d_{nk} z^n, \quad z \in S_k \tag{2.3}$$

where

$$a_k = 0, \quad k = 2, 3; \quad \kappa_1 = \kappa_2 = \kappa, \quad \kappa_3 = \kappa_4 = \kappa_0; \quad c_{12} = \Gamma, \quad d_{12} = \Gamma'$$

$$c_{n2} = d_{n2} = 0, \quad n = 2, 3, \ldots; \quad c_{n3} = d_{n3} = 0, \quad n = -1, -2, \ldots$$

Since the complex potentials $\varphi_k(z)$ and $\psi_k(z)$ are defined up to complex additive terms c_k and $\kappa_k \bar{c}_k$, respectively, it follows that the free terms in series (2.3) may be assumed, without loss of generality, to be equal to zero

$$c_{0k} = 0, \quad d_{0k} = \kappa_k \overline{c_{0k}} = 0, \quad k = 1, 2, 3, 4$$

On the assumption that the series (2.3) and the series obtained from them by differentiating term by term are uniformly convergent in the respective domains S_k, including their boundaries, they may be substituted into conditions (2.1), and to determine the remaining unknown coefficients a_k, c_{nk} and d_{nk} of the series we obtain an infinite system of linear algebraic equations, which may be divided into the following finite systems for the different groups of coefficients.

System 1

$$\mu_*(\kappa c_{11}R_1-\overline{c_{11}}R_1-\overline{d_{-11}}R_1^{-1}) = \kappa_0 c_{13}R_1-\overline{c_{13}}R_1 = \kappa_0 c_{14}R_1-\overline{c_{14}}R_1-\overline{d_{-14}}R_1^{-1}$$

$$h_*^{-1}(c_{11}R_1+\overline{c_{11}}R_1+\overline{d_{-11}}R_1^{-1})+c_{14}R_1+\overline{c_{14}}R_1+\overline{d_{-14}}R_1^{-1} = c_{13}R_1+\overline{c_{13}}R_1$$

$$\kappa c_{11}R_2-\overline{c_{11}}R_2-\overline{d_{-11}}R_2^{-1} = \kappa\Gamma R_2-\bar{\Gamma}R_2-\overline{d_{-12}}R_2^{-1} = \mu_*^{-1}(\kappa_0 c_{14}R_2-\overline{c_{14}}R_2-\overline{d_{-14}}R_2^{-1})$$

$$c_{11}R_2+\overline{c_{11}}R_2+\overline{d_{-11}}R_2^{-1}+h_*(c_{14}R_2+\overline{c_{14}}R_2+\overline{d_{-14}}R_2^{-1}) = \Gamma R_2+\bar{\Gamma}R_2+\overline{d_{-12}}R_2^{-1}$$

System 2

$$\mu_*(\kappa c_{-11}R_1^{-1}-3\overline{c_{31}}R_1^3-\overline{d_{11}}R_1) = -3\overline{c_{33}}R_1^3-\overline{d_{13}}R_1 = \kappa_0 c_{-14}R_1^{-1}-3\overline{c_{34}}R_1^3-\overline{d_{14}}R_1$$

$$h_*^{-1}(c_{-11}R_1^{-1}+3\overline{c_{31}}R_1^3+\overline{d_{11}}R_1)+c_{-14}R_1^{-1}+3\overline{c_{34}}R_1^3+\overline{d_{14}}R_1 = 3\overline{c_{33}}R_1^3+\overline{d_{13}}R_1$$

$$\kappa c_{-11}R_2^{-1}-3\overline{c_{31}}R_2^3-\overline{d_{11}}R_2 = -\overline{\Gamma'}R_2+\kappa c_{-12}R_2^{-1} = \mu_*^{-1}(\kappa_0 c_{-14}R_2^{-1}-3\overline{c_{34}}R_2^3-\overline{d_{14}}R_2)$$

$$c_{-11}R_2^{-1}+3\overline{c_{31}}R_2^3+\overline{d_{11}}R_2+h_*(c_{-14}R_2^{-1}+3\overline{c_{34}}R_2^3+\overline{d_{14}}R_2) = c_{12}R_2^{-1}+\overline{\Gamma'}R_2$$

$$\mu_*(\kappa\overline{c_{31}}R_1^3+c_{-11}R_1^{-1}-d_{-31}R_1^{-3}) = \kappa_0\overline{c_{33}}R_1^3 = \kappa_0\overline{c_{34}}R_1^3+c_{-14}R_1^{-1}-d_{-34}R_1^{-3}$$

$$h_*^{-1}(\overline{c_{31}}R_1^3-c_{-11}R_1^{-1}+d_{-31}R_1^{-3})+\overline{c_{34}}R_1^3-c_{-14}R_1^{-1}+d_{-34}R_1^{-3} = \overline{c_{33}}R_1^3$$

$$\kappa\overline{c_{31}}R_2^3+c_{-11}R_2^{-1}-d_{-31}R_2^{-3} = c_{-12}R_2^{-1}-d_{-32}R_2^{-3} = \mu_*^{-1}(\kappa_0\overline{c_{34}}R_2^3+c_{-14}R_2^{-1}-d_{-34}R_2^{-3})$$

$$\overline{c_{31}}R_2^3-c_{-11}R_2^{-1}+d_{-31}R_2^{-3}+h_*(\overline{c_{34}}R_2^3-c_{-14}R_2^{-1}+d_{-34}R_2^{-3}) = -c_{-12}R_2^{-1}+d_{-32}R_2^{-3}$$

System 3

$$\mu_*(\kappa c_{21}R_1^2-\overline{a_1}-\overline{d_{-21}}R_1^{-2}) = \kappa_0 c_{23}R_1^2 = \kappa_0 c_{24}R_1^2-\overline{a_4}-\overline{d_{-24}}R_1^{-2}$$

$$h_*^{-1}(c_{21}R_1^2+\overline{a_1}+\overline{d_{-21}}R_1^{-2})+c_{24}R_1^2+\overline{a_4}+\overline{d_{-24}}R_1^{-2} = c_{23}R_1^2$$

$$\kappa c_{21}R_2^2-\overline{a_1}-\overline{d_{-21}}R_2^{-2} = -\overline{d_{-22}}R_2^{-2} = \mu_*^{-1}(\kappa_0 c_{24}R_2^2-\overline{a_4}-\overline{d_{-24}}R_2^{-2})$$

$$c_{21}R_2^2+\bar{a}_1+\overline{d_{-21}}R_2^{-2}+h_*(c_{24}R_2^2+\overline{a_4}+\overline{d_{-24}}R_2^{-2}) = \overline{d_{-22}}R_2^{-2}$$

$$\mu_*\left(\kappa\overline{a_1}\ln\frac{R_2}{R_1}-c_{21}(R_2^2-R_1^2)\right) = \kappa_0\overline{a_4}\ln\frac{R_2}{R_1}-c_{24}(R_2^2-R_1^2)$$

System 4

$$\mu_*(\kappa\overline{c_{-n1}}R_1^{-n}+(n+2)c_{(n+2)1}R_1^{n+2}-d_{n1}R_1^n) = -(n+2)c_{(n+2)3}R_1^{n+2}-d_{n3}R_1^n =$$
$$= \kappa_0\overline{c_{-n4}}R_1^{-n}-(n+2)c_{(n+2)4}R_1^{n+2}-d_{n4}R_1^n$$

$$h_*^{-1}(\overline{c_{-n1}}R_1^{-n}+(n+2)c_{(n+2)1}R_1^{n+2}+d_{n1}R_1^n)+\overline{c_{-n4}}R_1^{-n}+(n+2)c_{(n+2)4}R_1^{n+2}+$$
$$+d_{n4}R_1^n = (n+2)c_{(n+2)3}R_1^{n+2}+d_{n3}R_1^n$$

$$\kappa\overline{c_{-n1}}R_2^{-n} - (n+2)c_{(n+2)1}R_2^{n+2} - d_{n1}R_2^n = \kappa\overline{c_{-n2}}R_2^{-n} =$$
$$= \mu_*^{-1}(\kappa_0\overline{c_{-n4}}R_2^{-n} - (n+2)c_{(n+2)4}R_2^{n+2} - d_{n4}R_2^n)$$

$$\overline{c_{-n1}}R_2^{-n} + (n+2)c_{(n+2)1}R_2^{n+2} + d_{n1}R_2^n +$$
$$+ h_*(\overline{c_{-n4}}R_2^{-n} + (n+2)c_{(n+2)4}R_2^{n+2} + d_{n4}R_2^n) = \overline{c_{-n2}}R_2^n$$

$$\mu_*(\kappa c_{(n+2)1}R_1^{n+2} + n\overline{c_{-n1}}R_1^{-n} - \overline{d_{-(n+2)1}}R_1^{-n-2}) = \kappa_0 c_{(n+2)3}R_1^{n+2} =$$
$$= \kappa_0 c_{(n+2)4}R_1^{n+2} + n\overline{c_{-n4}}R_1^{-n} - \overline{d_{-(n+2)4}}R_1^{-n-2}$$

$$h_*^{-1}(c_{(n+2)1}R_1^{n+2} - n\overline{c_{-n1}}R_1^{-n} + \overline{d_{-(n+2)1}}R_1^{-n-2}) + c_{(n+2)4}R_1^{n+2} - n\overline{c_{-n4}}R_1^{-n} +$$
$$+ \overline{d_{-(n+2)4}}R_1^{-n-2} = c_{(n+2)3}R_1^{n+2}$$

$$\kappa c_{(n+2)1}R_2^{n+2} + n\overline{c_{-n1}}R_2^{-n} - \overline{d_{-(n+2)1}}R_1^{-n-2} = n\overline{c_{-n2}}R_2^{-n} - \overline{d_{-(n+2)2}}R_2^{-n-2} =$$
$$= \mu_*^{-1}(\kappa_0 c_{(n+2)4}R_1^{n+2} + n\overline{c_{-n4}}R_1^{-n} - \overline{d_{-(n+2)4}}R_2^{-n-2})$$

$$c_{(n+2)1}R_2^{n+2} - n\overline{c_{-n1}}R_2^{-n} + \overline{d_{-(n+2)1}}R_2^{-n-2} +$$
$$+ h_*(c_{(n+2)4}R_2^{n+2} - n\overline{c_{-n4}}R_2^{-n} + \overline{d_{-(n+2)4}}R_2^{-n-2}) = -n\overline{c_{-n2}}R_2^{-n} + \overline{d_{-(n+2)2}}R_2^{-n-2}$$

$n \geq 2$

In addition, Eq. (2.2) should be added to System 3.

System 1–4 are always uniquely solvable, since the homogeneous systems obtained by putting $\sigma_x^\infty = \sigma_y^\infty = \tau_{xy}^\infty = \omega^\infty = 0$ have only trivial solutions – because the original mechanical system in that case has only a trivial solution. Hence it follows that System 3 and 4, which are homogeneous, have only trivial solutions, and the complex potentials may be represented in the form

$$\begin{aligned}
\varphi_1(z) &= c_{-11}z^{-1} + c_{11}z + c_{31}z^3, \quad \psi_1(z) = d_{-31}z^{-3} + d_{-11}z^{-1} + d_{11}z \\
\varphi_2(z) &= c_{-12}z^{-1} + \Gamma z, \quad \psi_2(z) = d_{-32}z^{-3} + d_{-12}z^{-1} + \Gamma' z \\
\varphi_3(z) &= c_{13}z + c_{33}z^3, \quad \psi_3(z) = d_{13}z \\
\varphi_4(z) &= c_{-14}z^{-1} + c_{14}z + c_{34}z^3, \quad \psi_4(z) = d_{-34}z^{-3} + d_{-14}z^{-1} + d_{14}z
\end{aligned} \tag{2.4}$$

The coefficients c_{jk} and d_{jk} are found from Systems 1 and 2.

This confirms our initial assumption, according to which the power series by which the complex potentials $\varphi_k(z)$ and $\psi_k(z)$ ($k = 1, 2, 3, 4$) are represented, as well as the series obtained by termwise differentiation of the series (2.3), converge uniformly in the respective domains S_k, including their boundaries.

3. INVESTIGATION OF THE STRESS STATE

By formulae (2.4) and the Kolosov–Muskhelishvili formulae [3], the stresses in the plate and the patch, expressed in polar coordinates r, θ at a point $z = re^{i\theta} \in S_k$, are found from the formulae

$$\sigma_r(z)_k = 2\mathrm{Re}c_{1k} + r^{-2}\mathrm{Re}d_{-1k} - (\mathrm{Re}d_{1k} + 4r^{-2}\mathrm{Re}c_{-1k} - 3r^{-4}\mathrm{Re}d_{-3k})\cos 2\theta +$$
$$+ (\mathrm{Im}d_{1k} - 4r^{-2}\mathrm{Im}c_{-1k} + 3r^{-4}\mathrm{Im}d_{-3k})\sin 2\theta$$

$$\begin{aligned}
\sigma_\theta(z)_k &= 2\mathrm{Re}c_{1k} - r^{-2}\mathrm{Re}d_{-1k} + (12r^2\mathrm{Re}c_{3k} + \mathrm{Re}d_{1k} - 3r^{-4}\mathrm{Re}d_{-3k})\cos 2\theta - \\
&- (12r^2\mathrm{Im}c_{3k} + \mathrm{Im}d_{1k} + 3r^{-4}\mathrm{Im}d_{-3k})\sin 2\theta
\end{aligned} \tag{3.1}$$

$$\tau_{r\theta}(z)_k = -r^{-2}\mathrm{Im}d_{1k} + (6r^2\mathrm{Im}c_{3k} + \mathrm{Im}d_{1k} + 2r^{-2}\mathrm{Im}c_{-1k} - 3r^{-4}\mathrm{Im}d_{-3k})\cos 2\theta +$$
$$+ (6r^2\mathrm{Re}c_{3k} + \mathrm{Re}d_{1k} - 2r^{-2}\mathrm{Re}c_{-1k} + 3r^{-4}\mathrm{Re}d_{-3k})\sin 2\theta; \quad k = 1, 2, 3, 4$$

where

$$c_{32} = c_{-13} = d_{-13} = d_{-33} = 0, \quad c_{12} = \Gamma, \quad d_{12} = \Gamma'$$

and the remaining coefficients c_{jk} and d_{jk} are found from Systems 1 and 2.

Let us determine at which point of the circle $|z| = r$ the stresses take extremal values. To that end, we rewrite formulae (3.1) as follows, omitting the arguments and indices in the stresses for greater convenience

$$\sigma_r = \alpha_1 + \mathrm{Re}(b_1 e^{2i\theta}), \quad \sigma_\theta = \alpha_2 + \mathrm{Re}(b_2 e^{2i\theta}), \quad \tau_{r\theta} = \alpha_3 + \mathrm{Im}(b_3 e^{2i\theta})$$

where α_1, α_2 and α_3 are real coefficients independent of the polar angle θ and

$$b_1 = -d_{1k} - 4r^{-2}\overline{c_{-1k}} + 3r^{-4}\overline{d_{-3k}}, \quad b_2 = 12r^2 c_{3k} + d_{1k} - 3r^{-4}\overline{d_{-3k}}$$

$$b_3 = 6r^2 c_{3k} + d_{1k} - 2r^{-2}\overline{c_{-1k}} + 3r^{-4}\overline{d_{-3k}}$$

Since

$$c_{32} = c_{-13} = d_{-33} = 0, \quad d_{12} = \Gamma'$$

and the remaining unknown constants c_{3k}, $\overline{c_{-1k}}$, d_{1k} and $\overline{d_{-3k}}$ may be found from System 2, where these unknown appear with real coefficients, it follows that $b_j = \beta_j \Gamma'$, where β_j are certain real numbers. Consequently

$$\sigma_r = \alpha_1 + \beta_1 |\Gamma'| \mathrm{Re}e^{i(\arg\Gamma' + 2\theta)}, \quad \sigma_\theta = \alpha_2 + \beta_2 |\Gamma'| \mathrm{Re}e^{i(\arg\Gamma' + 2\theta)}$$

$$\tau_{r\theta} = \alpha_3 + \beta_3 |\Gamma'| \mathrm{Im}e^{i(\arg\Gamma' + 2\theta)}$$

and the stresses σ_r and σ_θ attain their extremal values on the circle $|z| = r$ at polar angles $\theta_1 = -(\arg\Gamma')/2$ and $\theta_2 = (\pi - \arg\Gamma')/2$; the stress $\tau_{r\theta}$ attains its extremal values at $\theta_3 = (\pi - 2\arg\Gamma')/4$ and $\theta_4 = -(\pi + 2\arg\Gamma')/4$.

Thus, on each circle $|z| = r$ the extremal values of the stresses are attained at points that have the same polar angles θ_1, θ_2 or θ_3, θ_4, which depend neither on the polar radius of these points nor on the elastic and geometric parameters of the plate and the patch; they depend only on $\arg\Gamma'$, that is, on the force parameters which act at infinity. Now, in order to determine the extremal values of the stresses in each of the domains S_k, one must put $\theta = \theta_1$, $\theta = \theta_2$ or $\theta = \theta_3$, $\theta = \theta_4$ in formulae (3.1) and, considering the resulting power functions of the polar radius r, find the extremal values as r varies between limits determined by S_k.

To determine the displacements of the points of the curves L_1 and L_2, we use the following formulae [3]

$$2\mu_0(u + iv)_k(z) = \kappa_0\varphi_k(z) - z\overline{\varphi_k'(z)} - \overline{\psi_k(z)}, \quad z \in S_k; \quad k = 3, 4$$

and formulae (2.4). After elementary reduction, we obtain

$$(u + iv)(R_1 e^{i\theta}) = \frac{R_1}{2\mu_0}(-(3\overline{c_{33}}R_1^2 + \overline{d_{13}})e^{-i\theta} + (\kappa_0 c_{13} - \overline{c_{13}})e^{i\theta} + \kappa_0 c_{33} R_1^2 e^{3i\theta})$$

$$(u + iv)(R_2 e^{i\theta}) = \frac{R_2}{2\mu_0}((\kappa_0 c_{-14} R_2^{-2} - 3\overline{c_{34}}R_2^2 - \overline{d_{14}})e^{-i\theta} +$$
$$+ (\kappa_0 c_{14} - \overline{c_{14}} - \overline{d_{-14}}R_2^{-2})e^{i\theta} + (\kappa_0 c_{34} R_2^2 + \overline{c_{-14}}R_2^{-2} - \overline{d_{-34}}R_2^{-4})e^{3i\theta})$$

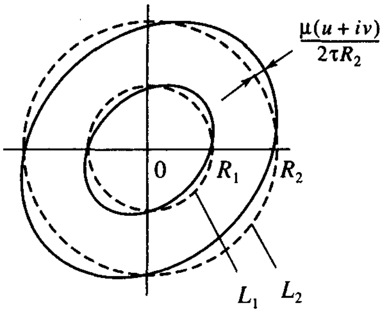

Fig. 1

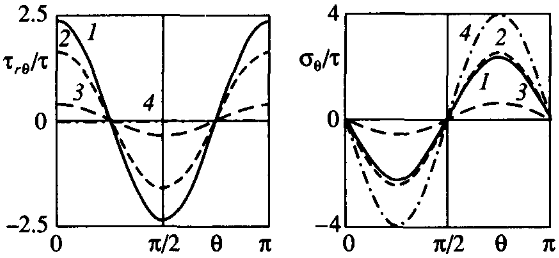

Fig. 2

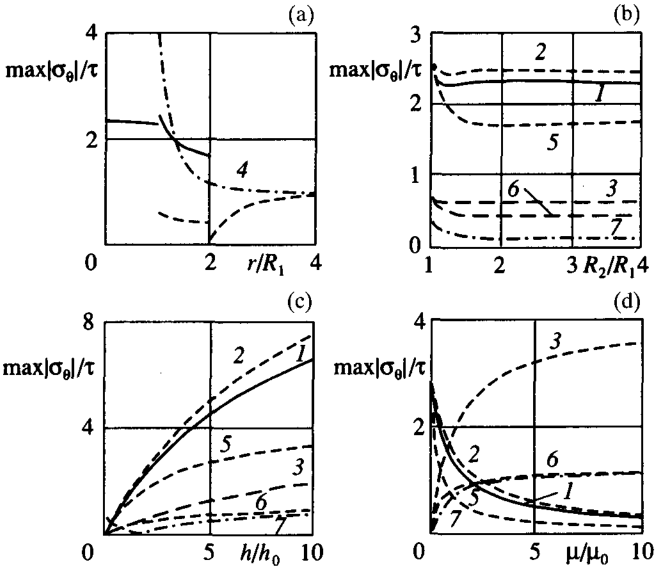

Fig. 3

Examples. Suppose the plate and the patch, the thickness of which is half that of the plate, have elasticity constants $\mu = 40$ MPa, $\nu = 0.37$ and $\mu_0 = 174.2$ MPa, $\nu_0 = 0.22$, corresponding to Cu–Al_2O_3 combination. The radii of the cut-out and the patch are in proportion as 1:2. Only a shearing stress $\tau_{xy}^{\infty} = \tau$ MPa (per unit thickness of the plate) is applied to the plate at infinity. All other initial force parameters are zero.

The solid curves in Fig. 1 represent the curves into which the boundaries of the cut-out L_1 and the patch L_2 are deformed, and the dashed curves are their initial positions before the load is applied. For greater clarity, the displacements of the points of the curves L_1 and L_2 are taken with the coefficient $\mu/(2\tau R_2)$.

In Fig. 2 we show graphs of the stresses $\tau_{r\theta}$ and σ_θ from inside and outside the attachment curves as viewed both form the plate and from the patch, as functions of the polar angle θ ($0 \le \theta \le \pi$). The stresses are symmetrically distributed on the lower half of the curve L_1 ($-\pi \le \theta \le 0$). Here and below the number 1 labels the graphs of the stresses on L_1 from inside as viewed from the plate, the numbers 2 and 3 label the graphs of the stresses on L_1 from outside as viewed from the patch and the plate, respectively, and number 4 labels the graphs of the stresses in the case of the classical problem of the stretching of a plate with a stress-free cut-out $|z| \le R_1$ subject to a distant load $\tau_{xy}^\infty = \tau$. The stress values in all the graphs are taken with coefficient τ^{-1}.

In Fig. 3 we show graphs of the stresses σ_θ of maximum magnitude in the plate and the patch, as functions of the quotient r/R_1 of the polar radius and the radius of the inner attachment curve (a), and graphs of the maximum of the same stress $|\sigma_\theta|$ on the attachment curves L_1 and L_2 as functions of the quotient of the radii of the patch and the initial cut-out (b), of the quotient h/h_0 of the thicknesses of the plate and patch (c), and of the quotient μ/μ_0 of the shear moduli of the plate and the patch (d). The solid curves in Fig. 3 correspond to the stress $\max|\sigma_\theta|/\tau$ in the patch, and the dashed lines to the stress in the plate. The new numbers 5 and 6 in the remaining parts of Fig. 3 label the graphs of the stress $\max|\sigma_\theta|/\tau$ on the curve L_2 from inside as viewed from the plate and the patch, respectively, and the number 7 labels the graphs of the stress on L_2 from outside as viewed from the plate. In all cases $\max|\sigma_\theta|$ is attained at points with polar angles $\theta = \pm\pi/4$ and $\theta = \pm 3\pi/4$. When there is no patch, $\max|\sigma_\theta|$ on the boundary L_1 of the stress-free cut-out $|z| \le R_1$ equals 4τ.

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